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On the Periodicities of Sunspots

Arthur Schuster

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II. *On the Periodicities of Sunspots.**

By ARTHUR SCHUSTER, *F.R.S.*

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1. THE material used in this investigation consists of :—

(*a*) The series of sunspot numbers collected by Dr. A. WOLF and Dr. A. WOLFER, and published by the latter in his "Astronomische Mitteilungen."† This series contains for each month of the years 1749 to 1901 the sunspot frequency as expressed in WOLF'S "Relativzahlen."

(*b*) The mean daily areas during each synodic revolution of the sun from January, 1832, to the end of 1900, which have been collected and are issued as a separate publication by the Solar Physics Committee. The publications of the Greenwich Observatory allow us to continue this series to the end of 1902.

(*c*) The Greenwich observations extending from 1883 to 1902, and giving for each day of the year complete statistics of the position and areas of spots. I have used the numbers representing the sums of the daily areas of the whole spots (including umbra and penumbra), the areas being corrected for fore shortening. The unit of area throughout this investigation is the millionth of the sun's hemisphere.

The three series will be referred to as (*a*), (*b*) and (*c*) respectively.

2. The series (*a*) being expressed in different units, the results obtained from it cannot be brought into very accurate comparison with the calculations based on the two other series. Identical units were not necessary for my purpose, but nevertheless it was advisable to investigate in how far a constant ratio existed between the sunspot frequency as expressed in WOLF and WOLFER'S series and the same frequency as more accurately determined by means of the measured areas. In WOLF'S sunspot numbers, each group of spots is weighted by a number which depends on the average number of spots in the groups visible at the time. If f is the total number of separate spots and g the number of groups, WOLF'S "Relativzahl" r is

$$r = \kappa\,(f+10g) = \kappa g\,(10+n),$$

* Preliminary notice published in 'Roy. Soc. Proc.,' A, vol. 77, p. 141.

† 'Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich,' vol. 47 (1902).

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if n is the average number of spots in a group. The factor κ depends on the instrument through which observations are conducted, and is taken to be unity for a telescope similar to the one used by WOLF.

In Table I.* I give in separate columns for each year the mean daily sunspot area, as published in series (b), together with the mean annual value of WOLF's numbers. The last column, giving the ratio between the two estimates of sunspot frequency, shows a discontinuity about the year 1860, the ratios after that date being markedly higher than those before it. The discontinuity seems to coincide with the time at which the areas began to be measured from photographs. I have therefore separately deduced the ratios for the different portions into which the complete series is divided. It is explained in publication (b), that SCHWABE's drawings were used up to the end of 1853, and from that date to the end of 1860 CARRINGTON's drawings were made use of in the reductions. For the year 1861 the gap between CARRINGTON's series and the Kew series of photographs had to be filled by means of SCHWABE's drawings. The Kew series reaches into the early part of 1872, and as all measurements of areas from 1832 to 1872 were made under the direction of DE LA RUE and STEWART, we may conclude that there is no great amount of personal equation so far as measurement is concerned. The Greenwich series begins in July, 1873, the gap intervening between the Kew and Greenwich observations being filled by measurements of the Wilna photographs supplied by Professsor BACKLUND.

Table II. shows the manner in which the ratio of WOLF's numbers to the sunspot areas has varied according to the material on which the measurements were based.

TABLE II.

Series.	Number of years.	Mean area divided by WOLF's numbers.
SCHWABE, 1832–1853	22	9·71
SCHWABE, including 1861	23	10·00
CARRINGTON, 1854–1860	7	10·79
DE LA RUE, 1862–1871.	10	16·92
Greenwich, 1874–1901	28	13·38
Total, 1832–1901	70	12·50

The last row gives the mean values as found from Table I.

The larger comparative value obtained when the areas are obtained from photographs is clearly shown by the table. It would seem therefore that WOLF's series as a whole is more homogeneous than that obtained from the measurement of areas. A second fact brought out by Table I. is that the ratios in the Greenwich series are markedly higher at times of sunspot maxima. This shows that in WOLF's method of

* Tables I., IV., V., VI., VII. and VIII. will be found at the end of the paper.

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counting, the sunspot activity is over-estimated when the activity is small, or under-estimated when it is large. The direct measurement of areas derived from photographic records gives therefore a greater range of variability between the minimum and the maximum. A further systematic difference will be pointed out in § 13. As it was only necessary for my purpose to bring the different series into approximate coincidence, I have used the mean value of the ratios and multiplied WOLF's numbers throughout by 12·5. This factor, at any rate as regards order of magnitude, reduces them to the scale of sunspot areas.

3. The main object of this investigation was to determine the periodogram of sunspot variability. Let a function $\phi(t)$ of the time t take the values ϕ_0, ϕ_1, ϕ_2, &c., at equidistant values of the time t_0, $t_0+\alpha$, $t_0+2\alpha$, &c., and let it be required to calculate the periodogram of $\phi(t)$ by means of the separate equidistant values.

Put

$$A = \sum_{s=0}^{s=(n-1)\alpha} \phi_s \cos \frac{2\pi}{n} s; \quad B = \sum_{s=0}^{s=(n-1)\alpha} \phi_s \sin \frac{2\pi}{n} s,$$

where n and s are integer numbers, and let

$$S = (A^2+B^2)\, \alpha/p.$$

Then by definition* the average value of S in the neighbourhood of a particular period $n\alpha$ gives the value of the periodogram for that period as derived from the time interval $p\alpha$. The different periods obtained by varying $n\alpha$ should be chosen near together, but there is a limit beyond which it would be useless to go. This limit is reached when the values of A and B for two closely adjoining values n_1 and n_2 are no longer independent of each other. The theory of vibration shows that independence begins when there is an ultimate disagreement of phase amounting to about one quarter of a period, so that if T_1 and T_2 are the times of two periods and the total number of periods is N, independence begins when

$$N(T_1 - T_2) = \tfrac{1}{4}T,$$

or when $T_1 - T_2$ is equal to T/4N, T being the approximate value of T_1 and T_2.

4. The periodogram may be said to put the statistical material in a form in which it may be most readily discussed, but there may always be cases in which the interpretation is difficult. A few words on this point may therefore prove useful. I do not, of course, claim to have first introduced the application of FOURIER's Theorem to the discovery of hidden periodicities. HORNSTEIN among many others made use of the harmonic analysis, and obtained for the elements of magnetic variations the Fourier coefficients for periods in the neighbourhood of 26 days. The process is sufficiently obvious to have been frequently introduced, but it has generally been assumed that each maximum in the amplitude of a harmonic term corresponded to a

* 'Roy. Soc. Proc.,' A, vol. 77, p. 136 (1905).

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true periodicity. The extent to which this fallacious reasoning has been made use of would surprise anyone not familiar with the literature of the subject.

What distinguishes the method which I am endeavouring to introduce from that of others, is the discussion of the natural variability of the Fourier coefficients according to the theory of probability, independently of any periodic cause which may have influenced the phenomenon. I have shown that where the phenomena are detached, and the probability of the occurrence of any one event does not depend on the occurrence of a previous one, there is a definite probability for the value of the amplitudes of the harmonic terms into which the recurrence of the phenomenon can be resolved. In the more complicated and more frequently occurring cases such a definite probability cannot be assigned *à priori*, but must be determined statistically from the phenomenon itself. Yet there is even in these cases a definite law which defines the manner in which the true periodicity gradually separates itself from accidental variations. The periodogram itself therefore furnishes the material for its discussion, but it is necessary that the investigation should be carried out systematically and extend over a large number of periods. If the statistical data spread over a sufficient range of time, it will often be found convenient to divide the total interval, so as to discover whether the periodogram as determined from one half is similar to that as determined by the other half. Other examples are given in the succeeding investigation illustrating the manner in which a comparatively scanty material may be used so as to give the greatest possible information.

I have so frequently insisted on the optical analogy that it may be worth while to point out that in one respect the periodogram furnishes more definite information than the optical instrument can. The spectroscope only determines the average "intensity," but the periodogram is also able to fix the phase of a periodic variation. Thus if the rainfall were analysed, the periodogram would show a maximum corresponding to the annual variation. With this maximum we may associate the angle determining the phase which will give us the date at which the maximum of the annual period takes place.

In fixing the phase of a periodicity, some care is necessary if the trial period is not exactly coincident with the true period, as otherwise an appreciable error may be introduced. It is necessary to discuss this point in some detail.

If in the expression of § 3 we write $\tan\phi$ for B/A, the angle ϕ measures the phase of the periodicity, supposing there is a true period having a time $T = n\alpha$. But if the true period is T′, the angle ϕ does not correctly represent the phase. To determine the error we may use the equations which I gave in the paper published in the Stokes Volume of the 'Cambridge Philosophical Transactions,' but the following short cut gives substantially the correct result. If we try to fix a simple periodic curve of period T, so as to make it most nearly agree with a periodic curve of period T′ during an interval varying from τ to $\tau + nT$, we must adjust their phases so that the extreme disagreement is as small as possible, and this is done by bringing the curves to

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coincidence at the midway point as regards time, *i.e.*, in the present case at a time $\tau+\frac{1}{2}n\mathrm{T}$.

We begin always with the same epoch, but provide for the possibility of subdividing the time interval so as to obtain separately the periodogram of different parts. We therefore must not neglect τ, but we may put it equal to an integral multiple of the trial period, or say to $m\mathrm{T}$. If NT is the total time interval considered, we have thus $n+m = \mathrm{N}$, so that the time at which coincidence of phase is required is $\frac{1}{2}(\mathrm{N}+m)\mathrm{T}$. If now $\cos(gt-\epsilon)$ is the true periodicity and $\cos(\kappa t-\phi)$ that of the trial period which is to be adjusted so as to fit the true period most nearly, we must put

$$\tfrac{1}{2}g(\mathrm{N}+m)\mathrm{T}-\epsilon = \tfrac{1}{2}\kappa(\mathrm{N}+m)\mathrm{T}-\phi,$$

therefore

$$\epsilon-\phi = \tfrac{1}{2}(g-\kappa)(\mathrm{N}+m)\mathrm{T} = \pi\frac{\mathrm{T}-\mathrm{T}'}{\mathrm{T}'}(\mathrm{N}+m).$$

This determines the difference in phase between that belonging to the true period and that determined with a slightly erroneous trial period. Table III. gives, *e.g.*, the angle ϕ in the neighbourhood of the 11-year period, and shows how the phase of the true periodicity may be determined. This periodicity, it will appear, is only prominent in the second portion of the complete time interval considered. The first column gives the times of the trial periods in years. The second and third columns give the values of N and m; in the fourth column is given the phase as calculated by the trial period, the fifth gives $(\epsilon-\phi)$ the difference of the phases between the trial and true periods calculated by means of the above equation on the supposition that the true period was 11·125 years. The last column gives the calculated phase of the true period. In calculating the last column from the two preceding ones it must be borne in mind that 360° or any multiple thereof may be added or subtracted from an angle representing a phase without alteration of its significant value.

TABLE III.

T.	N.	m.	ϕ.	$\epsilon-\phi$.	ϵ.
9·75	15	7	107°	−129°	−22°
10·00	15	7	35	41	− 6
10·25	14	7	306	297	+ 9
10·50	14	7	221	212	9
10·75	14	7	142	127	15
11·00	14	7	50	− 42	8
11·25	13	6	335	+ 38	13
11·50	13	6	250	115	5
11·75	13	6	180	192	12
12·00	12	6	107	255	2
12·25	12	6	37	328	+ 5
12·50	12	6	317	401	− 2
12·75	11	5	259	+421	−40

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The table shows in a remarkable manner how, quite independently of the intensity, the 11-year period ruled the periodogram during the interval considered. The phases ϕ are in regular progression, the intervals between two succeeding ones being almost equal. The last column gives the phase of the true period. As the calculation is only strictly applicable to periods near the real one, we may take 10° as the phase of the period as determined from the table. This gives for the time of the maximum $11{\cdot}125 \times \frac{10}{360} = 0{\cdot}31$ years, after the epoch which was 1749·25. The times of the other maxima are therefore included in the expression $1749{\cdot}56 \pm 11{\cdot}125n$, or, as we may also put it, $1905{\cdot}31 \pm 11{\cdot}125n$, where n is an integer number.

5. In order to obtain the periodogram for periods having a length of several years, it is obviously necessary to use as long a series as possible. I have consequently had recourse to WOLF's and WOLFER's monthly numbers, beginning with the year 1749. I have used the numbers given by WOLFER as "observed," and not those obtained by a process of smoothing, which are given in a separate table under the heading of "compensated numbers." The process of smoothing I consider to be a harmful process, and quite unnecessary when periodicities are to be examined by a scientific process.

Each year was divided into two equal parts, and the sums of the numbers for each half-year were calculated. These sums represented therefore six times the mean sunspot activity for the first and second halves of the year respectively. The values of A and B in the above expressions were then found. For this purpose the six-monthly values were, as is customary, arranged in rows containing as many figures as there are intervals of six months in the selected period. Thus, to get the value for twelve years, each row contained 24 numbers. The number of rows was arranged so as to get in as many complete periods as possible, beginning always with the total sunspot number for the first six months of 1749. We may therefore consider April 1, 1749, to be the epoch which determines the phase of all the periodicities derived from the series (a).

As twelve periods of twelve years bring us to 1894, there was not sufficient material to include a further complete period, and the calculations had to be stopped at that point. The sum of the twelve rows of 24 columns, in the example chosen, having been formed, the expressions for A and B are easily calculated by substituting $p = 288$ (which is twice the total number of years included). The year being the unit, and the intervals between the times to which successive numbers refer being six months, we must put $\alpha = 0{\cdot}5$. The ordinate of the periodogram will then be $(A^2+B^2)/144$. The factor in the denominator varies slightly for different periods, and sufficient accuracy is obtained by taking the divisor to be 150 in all cases.

As the sum of six successive sunspot numbers was used in place of the average during the corresponding interval, the result must be divided by 36 in order to bring

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it to WOLF'S scale, and multiplied by the square of 12·53* or 157, in order finally to reduce the result to the unit based on the measurement of areas. The method is available for all periodicities containing an integer multiple of the half year. In some cases it was advisable to obtain the periodogram for periods lying halfway between those to which the above method is directly applicable. The successive rows in the original table were then arranged so as alternately to contain a number of figures differing by one. Thus, for a period of $7\frac{3}{4}$ years the alternate rows were formed of 15 and 16 figures. This gives 31 intervals of six months for two complete periods, or on the average $7\frac{3}{4}$ years. In the last column alternate numbers were missing, and this column was omitted in the calculations for A and B, the number n being chosen to correspond to the number of columns retained. In some cases it is more convenient for the tabulation of the angles and of their trigonometrical functions to include the last column, which must in that case be multiplied by a factor correcting for its smaller number of entries. When a certain period has been treated, and the half or the third of that period is to be introduced, it is not of course necessary to form again the system of rows and columns, but we may proceed as in the calculation of the sub-periods of FOURIER'S coefficients. The equations are then replaced by

$$A = \sum_{s=0}^{s=(n-1)a} \phi_s \cos \frac{2m\pi}{n} s\,; \quad B = \sum_{s=0}^{s=(n-1)a} \phi_s \sin \frac{2m\pi}{n} s,$$

where m is two or three, according as the half or the third of the original period is required. (See Table IV., p. 97.)

6. The second column of Table IV. contains the ordinates of the periodogram, calculated from the complete record in the manner described, and in fig. 1 the periodograph is drawn. The great intensity of the 11-year period is well shown, but the shape of the curve for periods between 10 and 11 years and some other features seemed to require further elucidation. I therefore divided the whole period of 150 years into two portions, which were nearly equal to each other. This could be done accurately when the original table contained an even number of rows, but when the number was uneven, so that the two portions could not be made exactly equal, the first portion was always taken to be the shorter one. In consequence, the epoch of the beginning of the second portion varies somewhat; the exact dates are given in column 5 of the table, the Roman numeral attached to the year indicating whether it is the first or second six-monthly value which forms the beginning of the group of periods. Where no such Roman numeral is given, the sunspot numbers for the whole year were used in the calculations.

For some purposes it would have been better to have chosen a fixed epoch for the

* The general mean during the years 1832–1901 inclusive as given in § 1 is 12·50. When this investigation was commenced the areas for the year 1901 were not available. The number given in the text above, which is the mean found when 1901 is excluded, was used in the calculations. The difference is of course insignificant.

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beginning of the second and more important part of the series, in spite of the greater labour of calculation which this would have involved. But it was necessary to be able to compare the phases of the oscillations which the two halves yield separately. On the whole the balance of advantage seemed to lie with the method adopted.

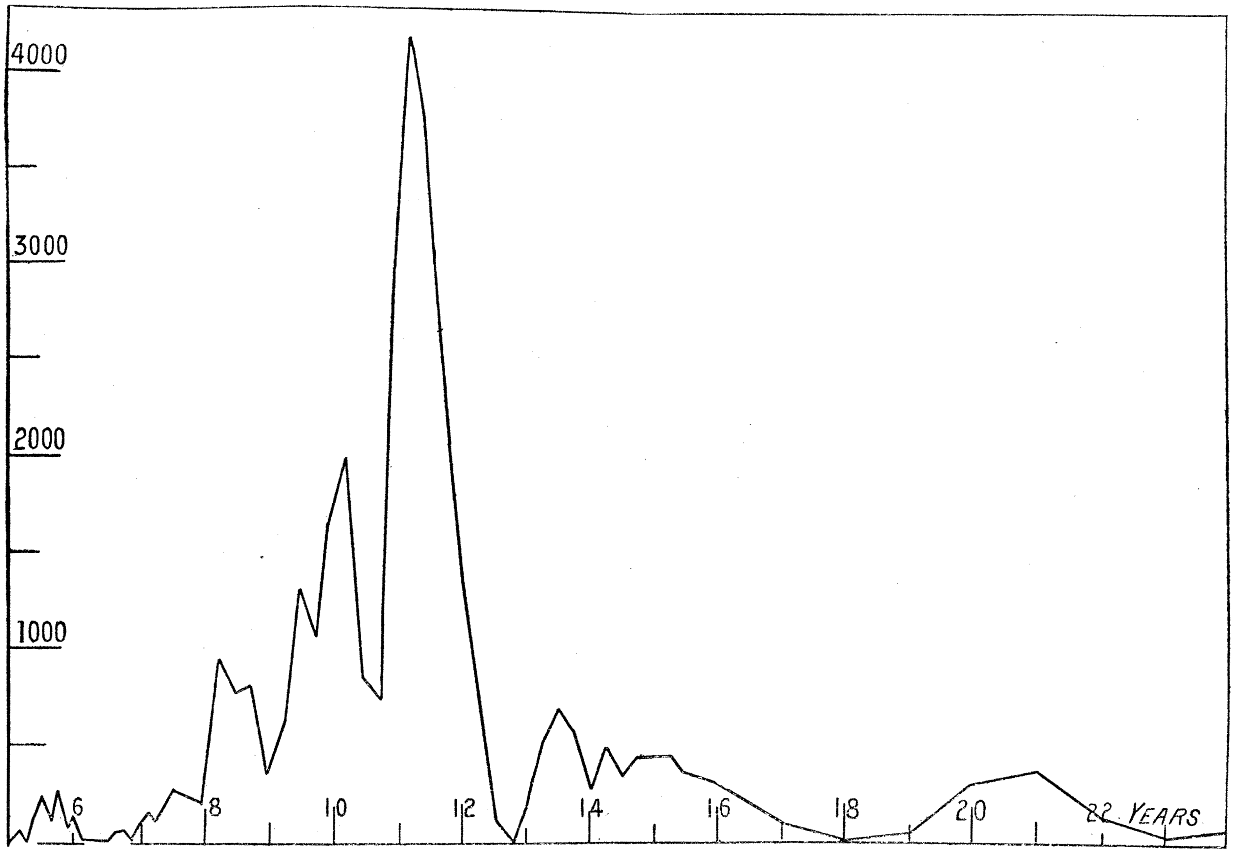

Fig. 1.

Columns 3 and 4 of Table IV. give the numbers obtained for the periodogram when the whole range is divided into two portions, and fig. 2 represents the numbers in a graphical form, the curve A being the periodograph belonging to the last 75 years.

The result is striking. The two curves have no resemblance to each other. Looking at them, we might think that they illustrate two entirely different phenomena. The truth is that between 1750 and 1820 the 11-year period, though existing, had a slight intensity, being quite overpowered by two others with periods of about $13\frac{3}{4}$ and $9\frac{1}{4}$ years. It may be argued that the observations during the eighteenth century were too scanty and uncertain to allow of any such definite conclusion, but I do not believe that the decisive verdict of the periodogram can be disposed of in this way. The main features of the curves are determined by the times at which the maxima occurred, and to a minor extent by the estimate of the activity at these maxima. The uncertainty which probably does exist near the times of the minima does not appreciably affect the result, as the periods of maximum activity as well

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as the intensities at these periods are sufficiently well known. There is, moreover, the direct evidence of the periodograph curve that the absence of the 11-year period is not likely to be due to accidental causes. Uncertainty in the observations would act in the direction of giving a broad band extending over a wide range of periods, but would not, except through a freak, give the two pronounced maxima which curve B shows.

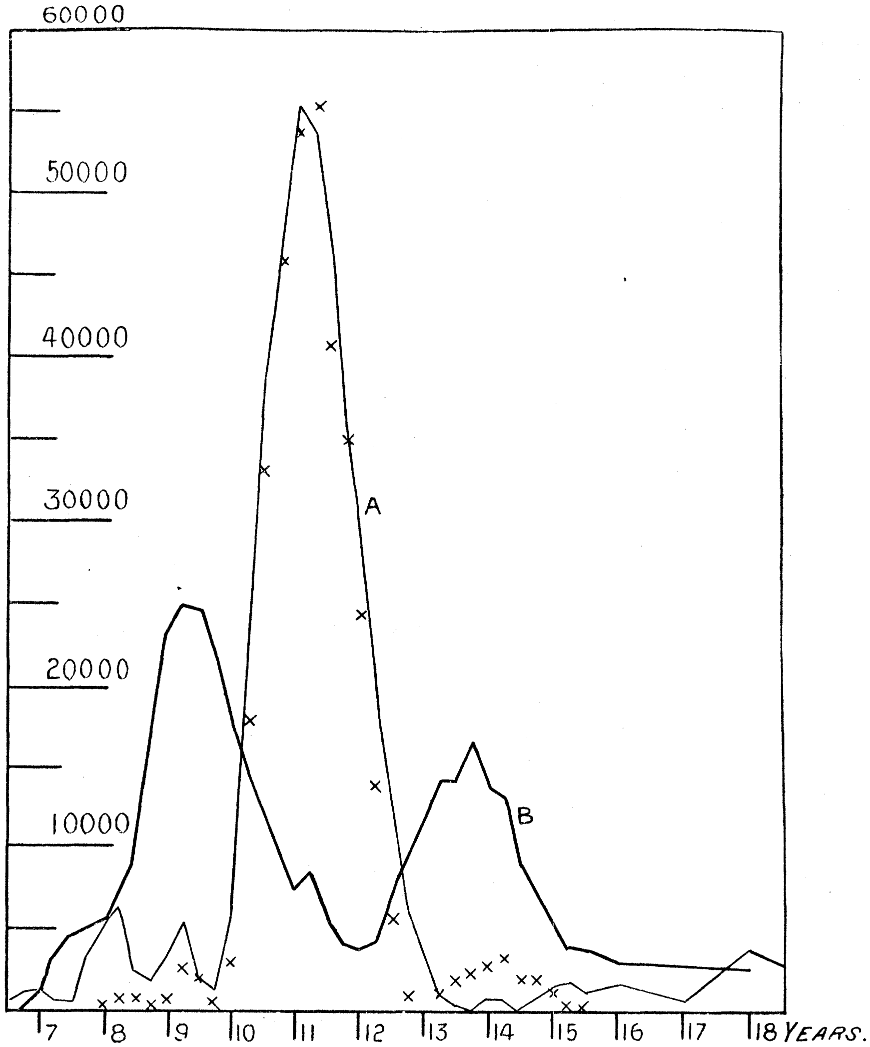

Fig. 2.

A closer inspection of the sunspot curves, as plotted down by Wolfer, shows that the two periods for which curve B shows its principal maxima were active successively rather than simultaneously. It appears that the period of 9 years is the important one as regards the observations previous to the maximum of 1788, while that of 13 and 14 years is brought in through the variations observed between 1788 and 1829. It would seem that more or less irregular variations, showing maxima of intervals of rather more than 9 years, were succeeded about 1788 by three unequal

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but long periods of 17·1, 11·2, and 13·5 years, respectively, and these afterwards settled down to a fairly regular periodicity of 11·1 years. It will appear that the periodicities in question are probably never quite extinct, but that the chief part in ruling the sunspot phenomena is sometimes taken up by one and sometimes by another period.

During the 75 years to which the curve A applies the shape of the periodogram is very nearly that which a true and, in the optical sense, homogeneous period would give. In order to show this, I have marked in fig. 2 by means of crosses the points through which the periodograph would pass if the period was a simple* one of 11·125 years. These crosses will be found to lie close to the actual curve. A curve drawn through the crosses, marking the periodograph of a homogeneous period, would show a small but appreciable rise corresponding to the periods of $9\frac{1}{4}$ and $14\frac{1}{4}$ years, and rather less marked elevations for shorter and longer periods. These secondary elevations correspond to the diffraction bands seen on the two sides of the central maximum of a spectroscopic line. It has already been pointed out in § 4 that the phases of the periodicities of nearly 11 years are in arithmetical progression such as we should expect if the main period were homogeneous.

Reference to Table IV. shows a decided maximum corresponding to a period of 5·625 years in the second half of the complete time interval. This is undoubtedly to a great extent due to the first sub-period of the 11·125-year cycle, though the displacement in time is greater than it should be. This maximum seems to exist to a much smaller degree in the first half of the interval. Further features of the table and curves will be referred to later.

The periodicity corresponding to the time of revolution of Jupiter (11·86 years) has been examined, but reference to Table IV. shows that there is no evidence of a periodicity connected with the orbital revolution of that planet. In fact, we may definitely assert that no influence directly traceable to Jupiter exists.

7. It has been stated that in the absence of definite periods the expectancy of the intensity of the periodogram must be obtained from the periodogram itself in all cases where the events to be analysed are not, as regards their succession, independent of each other. The expectancy not depending on the period, we may select for the purpose any portion of the curve in which we have no reason to suspect any periodicities. The portion most suitable for this purpose in our case is that lying between 54 days and 1·5 years. Shorter periods must be avoided, because if the length is only a few days the intensity of the periodogram is depressed, owing to the fact that sunspots as a rule last several days. That this is so may easily be recognised by imagining all the sunspots to have the same duration, each spot also keeping the same area during the whole course of its life. It is obvious that in this case the period having a length equal to the life of the spots would be totally absent.

* A simple period is one represented by a circular function.

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When we come to periods which are near to that of the solar rotation, periodicities appear, owing to the fact that some of the spots persist during more than one rotation. This effect will, however, disappear when the period is well above that of the solar rotation. When the periods come near to 1·5 years, the sub-periods of well-ascertained periodicities make their presence felt. Hence the limits chosen for calculating the natural intensity of the periodogram must be confined to about 35 days on the one hand and 1·5 years on the other. The average intensity of these periods may be determined from Table VI., and is found to be 15,000. The probability of an intensity greater than h times the average value is e^{-h}, and we may perhaps begin to suspect a real periodicity when this value is 1 in 200. This gives 5·3 as the value of h and 80,000 as the smallest value of the intensity which invites further discussion. When h has the value 8, the probability of an intensity greater than h times the expectancy is 1 in 3,000 and we may begin to be more confident that there is some definite cause at work to bring up the periodogram to that value. The intensity in that case is 120,000. When h is 16, the chances of being misled by accident is only one in a million.

8. Periodic times between 5·18 years and about 2 years were investigated by taking as basis the mean sunspot areas for each assumed synodic rotation, as given in publication (*b*). For this purpose four successive numbers, as given in the tables published by the Solar Physics Committee, were added together, and rows were formed containing a number of successive entries. The longest period chosen was that containing seventeen times four, or 68, synodic rotations, each assumed to be of duration 27·275 days, giving a period of 5·18 years. The process was repeated by forming tables containing columns varying in number between 10 and 17, and intermediate periods were treated by the well-known process of shifting successive rows to the right or left. The semi-periods were treated in the usual manner. Table V. gives the result, which is graphically represented in fig. 3, curve A. The periodogram curves, as deduced from WOLF's sunspot number in the manners explained above, are shown in curves B and C, the former referring to the second, and the latter to the first half of the total range of 150 years.

The main features of curve A are formed by the elevations at 2·69, 3·78, 4·38 and 4·78 years. The first two of these elevations are in all probability only due to sub-periods, being the second and third harmonic of the 11·125 variation which we should expect to find at 2·78 and 3·71. The divergence between the observed and calculated periods is probably not more than may be accounted for by superposed irregularities. The rise at 4·38 does not occur on curve C and only to a slight extent on curve B, but the intensity as determined from the areas is sufficient to make the reality of the period probable. There is also a slight rise of doubtful reality at a period corresponding to 4·08.

9. With regard to periods which are shorter than 1·89 years, the labour of a complete investigation which would have to secure that no existing periodicity could escape notice would be very great. I have, therefore, for the present considered only

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some of the special periodicities, the existence of which has been suspected, leaving a more detailed investigation for future treatment. In order to find the average values of the periodogram within this region, the mean sunspot areas (publication b) were grouped so that the times of the period were 16, 18, 20, ... 30 solar rotations. Each group was divided into four parts, corresponding to a division of the whole material, extending from 1832 to 1900, into approximately equal intervals of time,

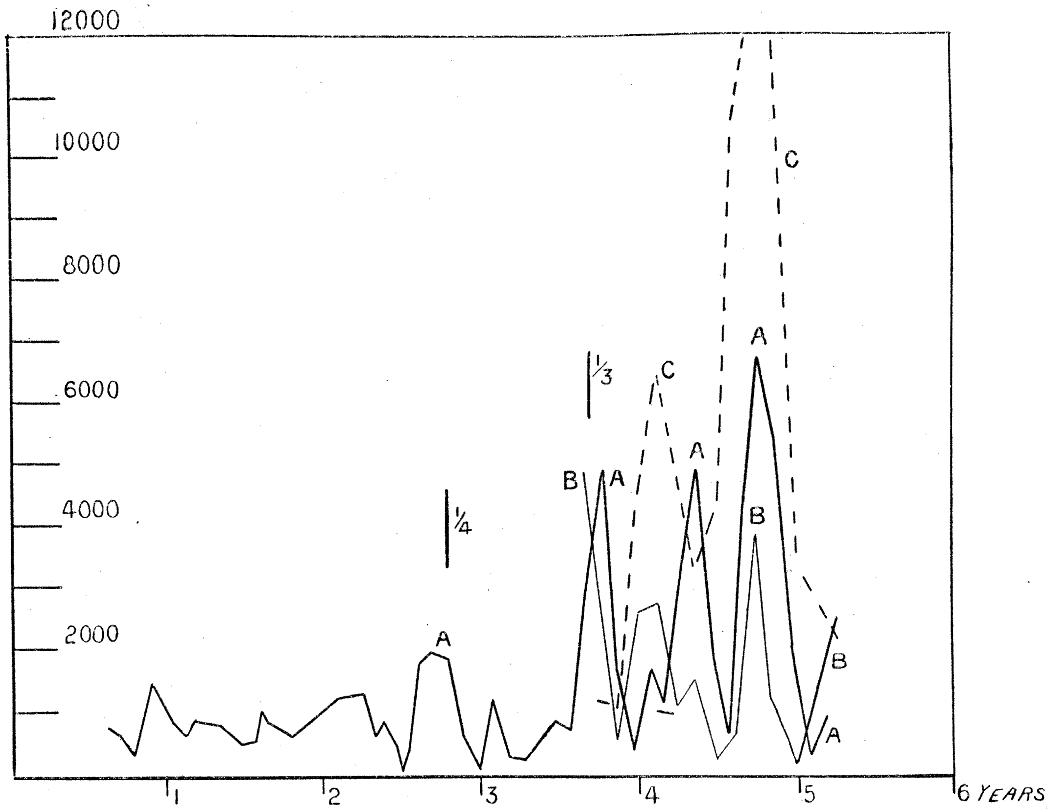

Fig. 3.

each such interval being therefore about $16\frac{1}{2}$ years. The mean intensity of the four groups for each period gives us a better estimate of the mean intensity of the periodogram than the values obtained from the group as a whole. If a periodicity were suspected, then of course the combined intensity of the whole time interval would have to be calculated. For the special investigation of a possible periodicity corresponding to the synodic revolution of Mercury, two tables were formed in which each row consisted of the numbers belonging to 4 and 5 rotations respectively, and each such table was subdivided into six groups. The calculations were completed by obtaining in each case the coefficients of the semi-periods. The results are given in Table VI.

For periods of still shorter duration the Greenwich results (c) had to be used. The available time interval, extending from 1886 to 1900, was divided into three sets of five years each, and periodic times of 24, 25 . . . . . 30 days were investigated,

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together with their first sub-periods. The intensities of the periodogram depend to a great extent on the particular part of the 11 years' cycle which is being investigated, the variations being greater when the intensities are great. The data are not sufficient to deal with a considerable number of cycles, and the correct average cannot be formed. I therefore give the values for each of the three groups 1886–90, 1891–95, and 1896–1900, the middle one of which coincides with a period of great activity. The numbers are collected in Table VII. There is a great variability in the numbers, but if a mean value is obtained for each period, and then, as in the last column of the table, an average of three successive periods is formed, a good agreement is obtained.

It will be noticed that somewhere between 15 and 24 days a considerable rise in the intensity of the periodogram takes place. This was to be expected. In the first place there must be a depression in the value of the intensities for short intervals of time due to the length of life of a spot. It has already been pointed out that if all spots lasted exactly the same time, the intensity of the periodogram would be zero for periodic times equal to this duration. The depression in the actual case will spread over a considerable range of periods, and will not allow the full value of the intensity to show itself until times which are at least twice as long as the average life of a spot. Such of the larger spots as come into view at one edge and traverse the whole disc would furnish a contribution to the periodogram in which the periods corresponding to half the synodic revolution are absent, provided the area covered by the spot is constant. We might therefore have expected a specially low value in the intensities corresponding to periods between 13·5 and 14 days. This is not shown by the tables, a fact probably accounted for by the great variability of the area of a large spot as it traverses the disc.

The periodic re-appearance of spots during successive rotations of the sun should increase the intensity for periods approximating to the average rotation period. A definite periodicity must not, however, be expected, quite apart from the fact that the rotation period itself is indefinite. The re-appearance of a phenomenon at regular intervals of time only constitutes an approximately homogeneous periodic phenomenon when the re-appearances are sufficiently numerous. With a life of three or four rotations only, there can be little distinction between periodicities ranging from 25 to 29 days.

The comparatively large value of the periodogram (2400) for a period of 13·25 days deserves some notice. Both in the interval 1890–95 and 1895–1900 there is a very decided rise of intensity corresponding to that period. But if the periodicity were real, the phases should be in regular progression. This is not the case, though we may obtain a better agreement by slightly altering the periodic time to 13·296 days. Treating the whole interval of fifteen years, the intensity is found to be 3080. If there were a real period, the increase should be much larger than this, and for the present we must therefore treat this period as accidental.

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The literature connected with the relation between solar terrestrial phenomena is full of supposed periodicities approximately coinciding with that of a solar rotation. Most of these periods are in the neighbourhood of 26 days, a time agreeing neither with the sidereal nor the synodic rotation, but it seemed worth while to investigate whether any definite periodicity in the sunspot areas can be discovered which takes place in a period near to that of 26 or 27 days. I have for this purpose worked out the Fourier coefficients of these two periods separately for each of the 15 years, 1885–1900. The deduced values of the periodogram and the phases are given in Table VIII. By a graphical method, explained in my discussion of the Magnetic Declination,* the periodicities of specially large intensity lying near the two selected times may be determined. Applying this method, I have not been able to discover any variation which suggests a true periodicity. The mean value of the periodogram for the periods of 26 and 27 days, as determined from individual years, agrees well with the general average shown by the last column of Table VII., which was based on a time range of 5 years.

10. A planetary influence on sunspots has often been suspected, and I have therefore specially considered the rotation periods of Jupiter, Mercury, and Venus. Table I. shows, as already pointed out, that the rotation period of Jupiter (11·86 years) gives no increased intensity to the periodogram. As regards the planets Venus and Mercury, Professor TURNER kindly sent me a list of the dates of their upper conjunction with the earth since 1833. Tables were formed in which, in the case of Venus, the values of the sunspot areas during 21 successive rotations were arranged in order, the first column always giving the sunspot area for the rotation during which upper conjunction took place. The synodic revolution of Venus being approximately 21·4 rotations, the tables so arranged could be used to determine the Fourier coefficients for the rotation period. In the case of Mercury the same process was followed, except that there were only four columns corresponding to the four complete solar rotations which take place within one synodic orbital revolution of the planet. Table VI. shows a small rise in the periodogram corresponding to the rotation period of 1·5986 years, which is that of Venus. The number given is the mean value of four, obtained by splitting the 68 available years into four periods of 17 years.

It is found that the phases for the four separate intervals do not agree, so that, if we combine the whole series of years, it is found that the periodogram intensity is smaller than the average. The synodic revolution of Mercury (115·88 days) similarly gives an exceptionally small value for the intensity of the periodogram. The semi-periods also show no effect.

11. We may now turn to the special consideration of the periodicities which have been found. The most persistent of the periods indicated by the periodogram is one having a period of about 4·8 years. It appears separately in the two series of WOLF's numbers and is confirmed by the records based on the measurement of areas.

* 'Cambridge Phil. Soc. Trans.,' vol. 18, p. 107 (1899).

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Appearing without much change in intensity throughout the time that sunspots have been observed, it has been more regular in its activity than the period of 11 years which hitherto has been the only one recognised.

To determine more exactly the periodic time, we may compare the phases as found from the two half intervals of Wolf's series, starting with the assumed period of 4·75 years. We thus find that there has been an acceleration of phase of 96° 43′ in the two portions which are separated by 16 complete periods. This change of phase is corrected by taking the periodic time to be 4·83 years. If we turn to the more accurate data derived from the measurements of areas, we find the maximum to lie at 4·78, but the change in intensity with increasing or diminishing periodic time suggests a somewhat longer period. A rough graphical interpolation seemed to give 4·81 years as the most probable length. Desiring to fix the time as accurately as possible, I arranged the records so as to give the intensity for this period and obtained 131,500, which is decidedly less than when a period of 4·78 years is assumed. This leaves us then with a certain amount of doubt as to the correct period, but the difference in the two values found would in 14 periods accumulate only to the extent of displacing the maximum by six months.

The phase as derived from the measurements of areas gives September 9, 1836, as one of the maxima, assuming the period to be 4·81 years, and September 21, 1836, if the period is 4·78 years. Wolf's numbers with a period of 4·83 years give February, 1836, as the corresponding date of maximum phase. Table IX. contains the dates of the maxima of the periods according to the two suppositions. The civil dates are only approximate and are derived from the fraction of the year. In comparing these dates in the future with the maxima as they occur, it will have to be noted that the phases given are those of the simple period, and that the higher harmonics have the effect of slightly postponing the maxima of the fundamental period.

Table IX.

4·81.	4·78.	4·81.	4·78.
1831·9 (November 17)	1831·9 (December 10)	1875·2 (March 3)	1875·0 (December 17, 1874)
1836·7 (September 9)	1836·7 (September 21)	1880·0 (December 23, 1879)	1879·7 (September 28)
1841·5 (July 1)	1841·5 (July 1)	1884·8 (October 15)	1884·5 (July 10)
1846·3 (April 23)	1846·3 (April 13)	1889·6 (August 7)	1889·3 (April 22)
1851·1 (February 13)	1851·1 (January 2)	1894·4 (May 29)	1894·1 (January 30)
1855·9 (December 5)	1855·8 (November 4)	1899·2 (March 21)	1898·9 (November 11)
1860·7 (September 27)	1860·6 (August 15)	1904·0 (January 11)	1903·6 (August 22)
1865·6 (July 19)	1865·4 (May 27)		
1870·4 (May 11)	1870·2 (March 4)		

The average amplitude of the period has been 98; that of the 11-year period being 586.

Perhaps the severest test we can apply to prove the reality of the period is to

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consider the individual maxima at times when otherwise the sunspot activity is small. The maxima of December 4, 1855, December 23, 1879, and August 7, 1889, satisfy that condition. According to the records tabulated by the Solar Physics Committee, the sunspot activity between March 15, 1855, and July, 1856, was exceedingly small, rising above 30 (the unit is the millionth of the sun's hemisphere) only in the two rotations beginning October 20 and November 16. During the latter rotation the area was 61. The only notable outbreak during 16 months, therefore, took place at a time closely coincident with the theoretical maximum. According to CARRINGTON's drawings the spots seen were close to the equator and in the northern hemisphere.

At the end of 1879 the sunspot activity began to increase again towards the maximum, which took place five years afterwards. The first notable outbreak took place during the rotation which began on December 30, closely following the maximum of the period we are considering.

The year 1889 was one of minimum activity, though spots were occasionally observed. The sunspot areas in the rotations beginning on July 21 were 345, the four preceding rotations gave 19, 17, 160, 214, and the four following ones 101, 59, 2, 0. There was here, therefore, a very sharp outbreak during an otherwise quiescent time. The outbreak occurred in the Southern hemisphere. There are other equally marked signs of increased activity at times which are close to all the calculated maxima, except that of 1860. This year coincides with a general maximum at which the main and most powerful influence which causes the 11-year period must only be expected to overshadow the weaker periodicity. The year 1894 was one in which the sunspot activity began to show a downward tendency. The most notable spots were noticed in the rotation beginning with May 29, the exact date of the calculated maximum of the period of 4·81 years. Here again the chief spots were in the Southern hemisphere. If we identify the outbreaks in the rotations beginning November 27, 1836, and March 10, 1899, with this particular period, the periodic time is found to be 4·79. I am inclined to think that the evidence of the more accurate observations in the last fifty years is in favour of a period somewhat less than 4·81, but further observations are needed to decide this point. Provisionally, I adopt 4·79 as the most probable time, and $1903{\cdot}72+4{\cdot}79n$ as the most probable date of future maxima.

Table X. may serve to facilitate the further study of this period. It gives the sum of the sixteen columns containing the entries from which the periodicity was deduced. In the original table each entry represented the sum of the mean areas during four successive rotations, and the numbers given here are the sum of fourteen rows, each row corresponding to a period of 4·78 years. The central date of the first entry which fixes the epoch is March 15, 1832, or 1832·203. The amplitude and phase are calculated from the values of $A = +41406$ and $B = -15249$. The phase is found to be $-20° \; 13'$, and this gives the date of the first maximum as $-\frac{20{\cdot}25}{360} \times 4{\cdot}75$, or 0·267 years before the epoch, *i.e.*, in 1831·94.

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TABLE X.

1.	2.	3.	4.	5.	6.	7.	8.
44,423	41,514	36,480	37,004	33,806	32,660	30,671	32,407
662	598	128	276	445	209	171	23
3,211	2,117	—	—	—	—	—	—
9.	10.	11.	12.	13.	14.	15.	16.
32,476	38,929	30,881	32,326	43,254	36,576	38,499	47,869
278	71	313	50	273	298	742	639

The rows to be added in future should be formed as follows: to the first number (44,423) add the sum of the mean daily areas of those four successive rotations, the first of which carries the number 605 in CARRINGTON'S series and begins on December 18, 1898. This number (662) and the following ones completing the first new row have been entered in the table, as well as the two first numbers belonging to the second row. In each new row the number of the first rotation which is taken into account is that included in the expression $605+64n$. If the new set of numbers is treated separately and a change of phase is found to occur, this would indicate a change in the periodic time. For the purpose of estimating the change, the above phase should be considered to belong to the mean point of the interval hitherto treated, *i.e.*, to June, 1865.

12. Although the period ot 4·38 does not rest on so secure a basis as the one we have just discussed, it has been very regular in its action during the last five cycles. There have been notable outbreaks of sunspot activity during the rotations 398 (July 4, 1883), 463 (May 11, 1888), 518 (June 19, 1892), 579 (January 8, 1897), and 637 (May 10, 1901), which seem to belong to this period. The maxima take place at dates which may be calculated from the expression $1897{\cdot}00 \pm 4{\cdot}38 \times n$, where n is an integer. Table XI. is constructed similarly to Table X., but as the period does not contain an integer number of four rotations, a certain adjustment in the succession of rows, to prevent a systematic shift of phase, had to be introduced. The row of numbers may, however, now be used to represent the periodicity of the indicated duration, and further rows should be added by beginning each of them

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with a sum of four rotations, the first of which coincides most nearly with the date $1897{\cdot}77 + 4{\cdot}38\,n$.

TABLE XI.

1.	2.	3.	4.	5.	6.	7.	8.
39,998	39,523	33,196	34,432	39,648	37,201	32,956	34,324
1,276	1,688	764	2,179	662	598	128	276
228	150	219	811	579	3,017	1,386	—
9.	10.	11.	12.	13.	14.	15.	—
37,989	41,414	45,278	49,487	42,504	43,543	38,944	—
445	209	171	23	278	71	313	—

The records which have been published since this investigation was first begun enable us to add more than one complete column to those which I have used, and these have been entered in the table.

13. The principal period which gives the present sunspot curve its characteristic shape deserves treatment in detail. Fig. 4 shows the average variation of the six sunspot cycles included between 1833–1899. The curve gives the rise and fall of the annual value of the mean daily area, while the crosses indicate the position of the points through which the curve would pass if deduced from WOLF's sunspot numbers. The close proximity of the crosses to the curve shows that, at any rate as far as the long-period variations are concerned, WOLF's numbers are proportional to the sunspot areas, and that the factor I have used is sufficiently accurate. The figure also illustrates the fact that WOLF's numbers underrate somewhat the activity near the times of a maximum, as has already been pointed out in § 1. But this deficiency is apparent throughout the rising branch of the curve. This points to a systematic difference in the sunspot statistics before and after the maximum. I have carefully examined the sunspot records as far back as the Greenwich records allow me to do so, and there seems indeed to have been a tendency during the 20 available years towards larger average areas (and also average duration) of spots during the rise as compared with the fall in the sunspot cycle. This would explain the deviations of WOLF's numbers from the curve marking the areas.

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The general character of the 11-year cycle not coming out with sufficient clearness when only annual averages are taken into account, I have deduced a more detailed curve (fig. 5), basing the calculation on the mean areas obtained during four successive rotations. We notice the sharp rise from the minimum, which takes place about $3\frac{1}{2}$ years before the maximum elevation is reached. The maximum here shows three distinct peaks, and although their position may vary in the individual maxima,

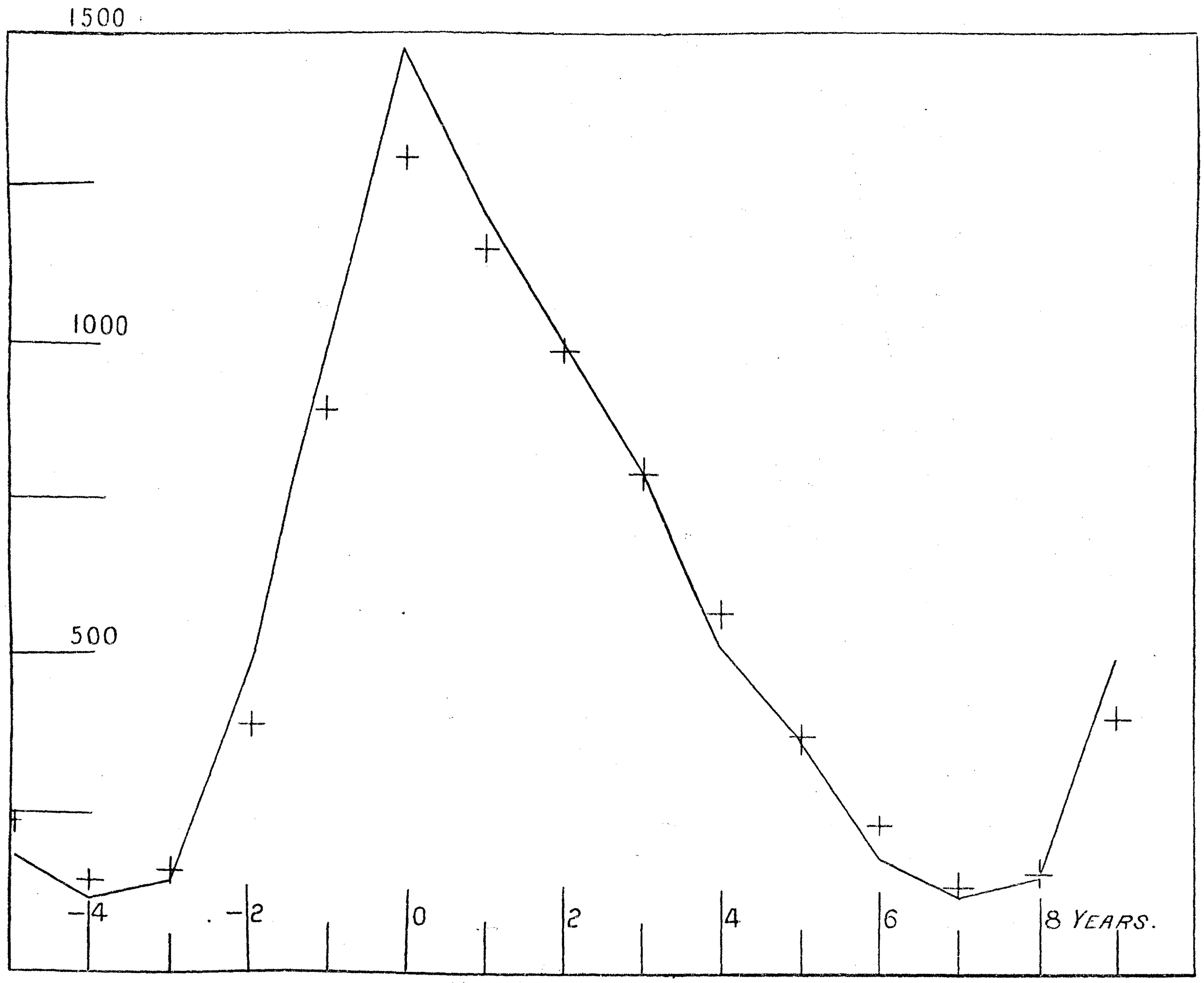

Fig. 4.

the sharp secondary rise and fall of the activity in the time immediately following a maximum is characteristic. The temporary recovery of activity about 3 years after the maximum is also a feature which may often be traced in the separate cycles.

I have investigated the sub-periods as well as the main 11-year period, taking the measurement of areas (publication *b*) as basis. Table XII. gives the results. It will be noted how closely the phase of the main period agrees with that derived in § 4 from WOLF's numbers.

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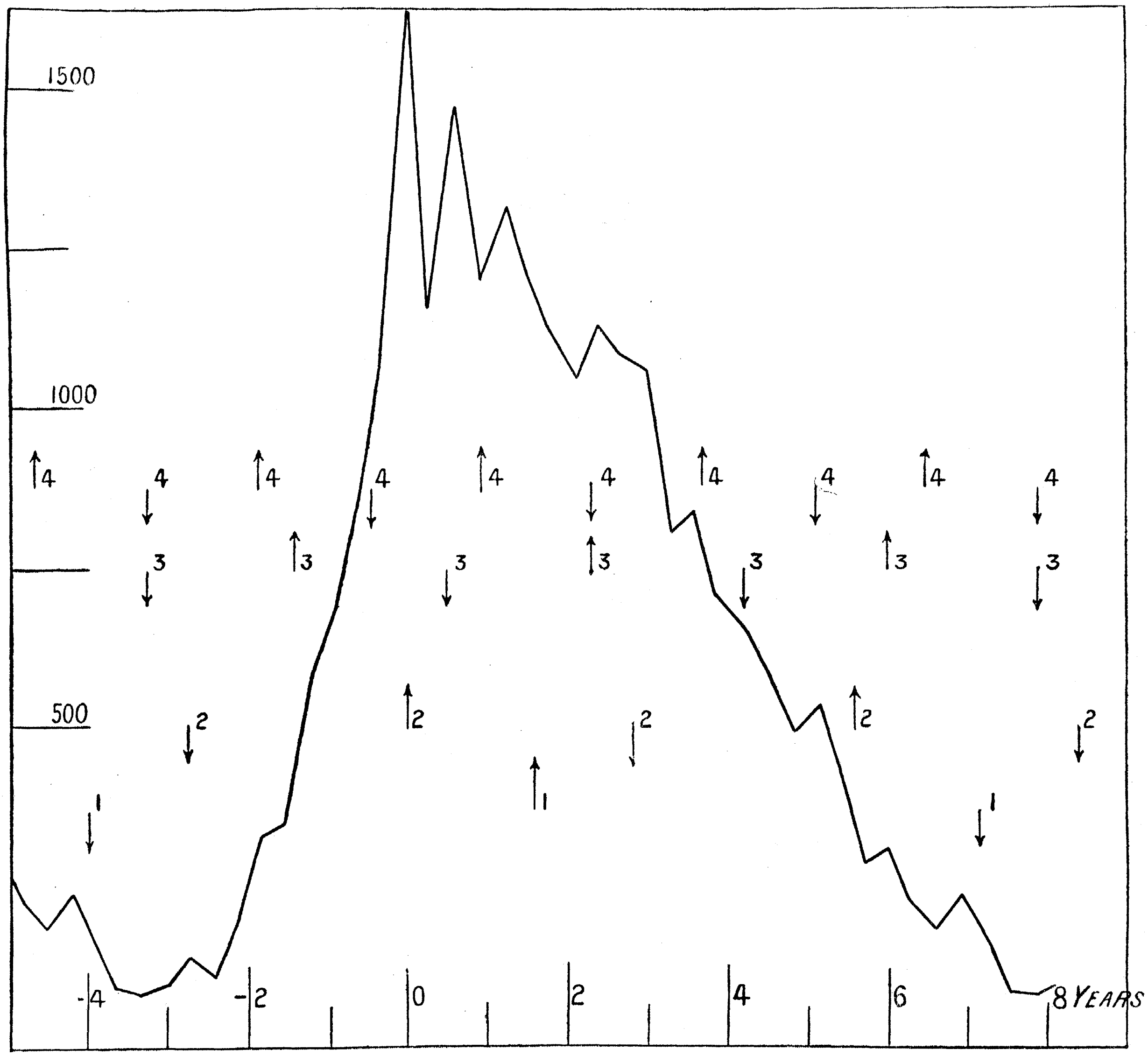

Fig. 5.

TABLE XII.

Period in years.	Amplitude (mean daily areas).	Ratios of Amplitudes.	Dates of maxima.
11·12	586	1	$1905 \cdot 35 \pm 11 \cdot 12 \times n$
$\frac{1}{2} \times 11 \cdot 12$	142	0·242	$1903 \cdot 75 \pm \frac{1}{2} \times 11 \cdot 12 \times n$
$\frac{1}{3} \times 11 \cdot 12$	64	0·109	$1902 \cdot 35 \pm \frac{1}{3} \times 11 \cdot 12 \times n$
$\frac{1}{4} \times 11 \cdot 12$	50	0·085	$1904 \cdot 66 \pm \frac{1}{4} \times 11 \cdot 12 \times n$

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The highest point of the complete curve is reached at a time $1903{\cdot}76 \pm n \times 11{\cdot}12$, but there is naturally a good deal of uncertainty as to the dates of the maxima, within the limits of a year. The positions of the maxima and minima of the harmonic components as given in Table XII. are marked in fig. 5 by an upward arrow denoting a maximum and a downward arrow a minimum. The order of the harmonic is indicated by the figures attached to the arrows. A retardation of 18 months in the principal maximum may occur owing to the temporary disappearance of the first harmonic, and does not therefore necessarily indicate an irregularity in the main periodicity.

The discussion of the 11·125 years' period has been entirely based on the results of the last 75 years of last century. Between 1749 and about 1826 the periodogram only gives a faint indication of the period. This remnant I ascribed at first without further investigation to the fact, that the 11 years' period had come into life towards the beginning of the century, and therefore had to some extent affected the interval 1750–1826 to which curve B of fig. 2 applies. This was an error which for some time kept me off the right track.

The question of the permanence of the periodicities is one of vital importance, and I have endeavoured throughout to keep my mind free from all preconceived notions. Every one thus looking with unprejudiced eyes at the main facts must feel himself drawn in opposite directions. On the one hand there is the evidence of the periodogram which indicates almost decisively a new departure at the beginning of the nineteenth century; on the other hand we have the remarkable fact that WOLFER, making use of all the data since 1610, deduced 11·124 years as the most probable period—a result identical with mine. Similarly NEWCOMB,* after a most careful examination of the same records, arrives at a period of 11·13 years. It needs to be explained why the years in which, according to our results, the periodicity was apparently absent do not affect the mean duration of the period, so that the same value is obtained whether the time during which the periodicity is *ex hypothesi* absent is included in the calculations or not. My first impression was that the coincidence was partly a matter of accident and partly due to the fact that both observers attached weights to the times of the observed maxima according to their trustworthiness. The weights thus attached to the records of the last 70 years naturally much exceeded those with which previous records were credited. Hence the whole result might very well be found to depend on the later periods only. When first writing out the account of my work, this was the view I took, and it was only very gradually that I abandoned it. I am convinced now that it is untenable.

To put the reader into possession of all the facts, and to get at a clear view of the subject, it seems advisable to state in detail the arguments which seemed to point to the conclusion that the 11 years' period was only of recent origin. In Table XIII. the first column contains the times of sunspot maxima as copied from WOLFER'S tables. The third column contains the intervals between two successive maxima. The fourth

* 'Astrophysical Journal,' vol. 13, p. 1 (1901).

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gives the sum of four successive intervals. Starting with the last figure so as to make use of all the best determined maxima, the first must be omitted when the intervals are collected in groups of four. It will be noticed that the average interval of four periods is approximately constant from 1675 to 1848, or during a period of over 170 years; the average of the periods which took place during that time is consistently about 10·8 years. It follows that, independently of the observations in the last 60 years, it is only the long intervals between 1626 and 1675 which bring up the average to 11 years. The latter part of this interval was one of great scarcity of sunspots, very few being observed between 1640 and 1670, and the whole of the second part of the seventeenth century seems to have been quite anomalous, as regards the behaviour of spots.*

TABLE XIII.

Maxima.	Δ.	Interval.	Sums of four successive intervals.	Maxima.	Δ.	Interval.	Sums of four successive intervals.
1615·5	−0·8	—	—	1761·5	+1·0	11·2	43·3
1626·0	−1·4	10·5	—	1769·7	−1·9	8·2	—
1639·5	+1·0	13·5	—	1778·4	−4·2	8·7	—
1649·0	−0·6	9·5	—	1788·1	−5·6	9·7	—
1660·0	−0·6	11·0	—	1805·2	+0·4	17·1	43·7
1675·0	+3·3	15·0	49·0	1816·4	+0·5	11·2	—
1685·0	+2·2	10·0	—	1829·9	+2·9	13·5	—
1693·0	−0·9	8·0	—	1837·2	−0·9	7·3	—
1705·5	+0·5	12·5	—	1848·1	−1·1	10·9	42·9
1718·2	+2·1	12·7	43·2	1860·1	−0·2	12·0	—
1727·5	+0·3	9·3	—	1870·6	−0·8	10·5	—
1738·7	+0·4	11·2	—	1883·9	+1·4	13·3	—
1750·3	+0·9	11·6	—	1894·1	+0·5	10·2	46·0

[The assumed grouping is admittedly arbitrary, but serves to show how difficult it is to devise a satisfactory measure of a period from the observations of maxima which are unequally spaced.—*February* 18, 1906.]

Assuming the permanency of the 11 years' period, both NEWCOMB and WOLFER formed tables giving the deviation in time of the observed maxima with those of a mean period derived from the whole of the available material. These deviations, according to WOLFER's estimate, are given in the second column of Table XIII. It will be seen at once that though differences occur amounting in one case to 5·6 years, yet the maxima seem on the whole to group themselves in pretty close proximity to the calculated times. Professor NEWCOMB sums up his conclusions in the sentence :—

"Underlying the periodic variations of spot activity there is a uniform cycle, unchanging from time to time, and determining the general mean of the activity."

A convincing feature of the second column of Table XIII. is that when maxima such

* According to SPORER there is no record of a sunspot being seen on the Northern Hemisphere of the Sun between 1672–1705.

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as those of 1727, 1738, 1750, and 1761 are at a distance of nearly 11 years, the differences between the calculated and observed times are always small. The two maxima of 1649 and 1660 fit in well in a similar manner with the computed times. These facts are difficult to explain unless we believe in the permanency of the period. Having thus once more become a partial convert to that belief, I turned again to the periodogram of 1750 to 1826 (curve B, fig. 2). The small peak showing at $11\frac{1}{4}$ years I had, as explained, ascribed to the years 1800–1826, during which, as I then thought, the 11-year period had come into existence. In order to clear up the matter, I investigated separately the five cycles between 1749 and 1794. The 11 years' oscillation now came out with an amplitude of 256, rather larger than I expected (the present amplitude being 586), and the phase was such as to give a maximum in 1747·5. A period of exactly $11\frac{1}{8}$ years would bring us after thirteen cycles to about 1903·1, agreeing to about two years with the actual date of the maximum of the 11-year period, as given in Table XII.

[Not much value can be attached to the phase calculated from five periods only at a time, when we know that other and stronger influences were at work. The conclusion to be drawn is that even when the periodicity in question was weak, its phase was not inconsistent with that calculated from other intervals of time.—*February* 18, 1906.]

The result reached at this point was that the 11 years' period though diminishing occasionally in intensity was yet permanent throughout the range of available sunspot records, and that whenever the time of successive maxima was nearly equal to 11 years they fitted in well with each other as regards phase.

14. Let us follow up the suggestion furnished by the last remark. Notice that during the three centuries to which Table XIII. applies there were three cases in which the interval between two successive sunspot maxima was nearly 13·5 years. Table XIV. shows that the intervals between each of these three single periods is nearly a multiple of the same duration, so that all figures fit in with a general periodicity of about 13·6 years.

TABLE XIV.

Sunspot maxima.	Length of period.	Intervals between periods.
1626 } 1639·5 }	13·5	
—	—	1816·4 – 1639·5 = 176·9 = 13 × 13·61.
1816·4 } 1829·9 }	13·5	
—	—	1870·6 – 1829·9 = 40·7 = 3 × 13·57.
1870·6 } 1883·9 }	13·3	
—	—	1883·9 – 1626 = 257·9 = 19 × 13·57.

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It should be noted that the first of the intervals of 13·5 years occurred previous to 1649, and is not brought into the calculation of the periodogram, which shows a decided maximum (fig. 1) for a period of 13·5 years. This period has evidently been one of the characteristic ones during the time to which curve B (fig. 2) applies. The fact that the maxima 1870·6 to 1883·9 are connected with it shows that it has still been active till quite lately. If it does not appear in curve A (fig. 2), this is accounted for by the fact that the first diffraction band of the 11 years' period overlaps and neutralises it. When the large period is eliminated, a rise at about 13 years becomes apparent. The process of elimination is, however, difficult, and errors may easily be introduced by means of it. There is evidence of two overlapping periods here, the broadness of the rise in curve B (fig. 2) being one of the indications of the duplicity. I refrain for the present, therefore, from discussing the period further.

15. Having found the average action of the $11\frac{1}{8}$ years' period, we may eliminate it from the general sunspot curve. The result is shown in fig. 6, in which the annual

Fig. 6.

values of sunspot areas since 1833 (publication *b*) has been used. The process of elimination brings out a period of about 8·5 years in a striking manner. The maxima occurred in the years 1836, 1845, 1853, 1862, and 1870. The periodogram for the combined interval 1749 to 1900 shows a maximum at the period of 8·25 years. Taking the phase as obtained by means of the 8·25 years' oscillation from the interval 1749 to 1826 alone, and forecasting the maxima for the subsequent interval, we obtain 1836·1, 1844·5, 1852·7, 1861·0, and 1869·2, in almost exact agreement with the above. In fig. 6 short lines are drawn indicating the maxima of the curve, while arrows mark the positions of the maxima calculated, as first explained, from the observations which are not included in the time scale of the figure.

The phases of the 8·25 years' period, as calculated from the two intervals separated by nine periods, are given in Table IV., and are there found to differ by 26°. When minutes of arc are taken into account, this angle is more accurately found to be 25° 16′. To bring the phases into coincidence, the period would have to be slightly lengthened, so as to be equal to 8·31 years. If we start from the 8·5 years' oscillation, we find similarly a diminution of phase equal to 34° 5′, and the period calculated so as to destroy this difference would be 8·41 years. We may adopt the mean of these results and take 8·36 as the most probable period. To get the phase we may use the method

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explained in § 4, which leads to the results given in Table XV. For each period we obtain three numbers according as we use the whole interval or the two separate halves.

TABLE XV.

Period.	Calculated phases of 8·36-year period.		
	1749–1900.	1749–1826.	1826–1900.
7·5	353°	210°	340°
7·75	—	215	199
8	191	196	173
8·25	198	207	190
8·5	217	201	221
8·75	278	229	140

It will be noticed that, except as regards the values derived from the shortest and longest periods included in the table, the phases are concordant. The amplitude of this period being only about one-third of that of the 11 years' period, it is more likely to be interfered with by accidental variations, and therefore a closer agreement

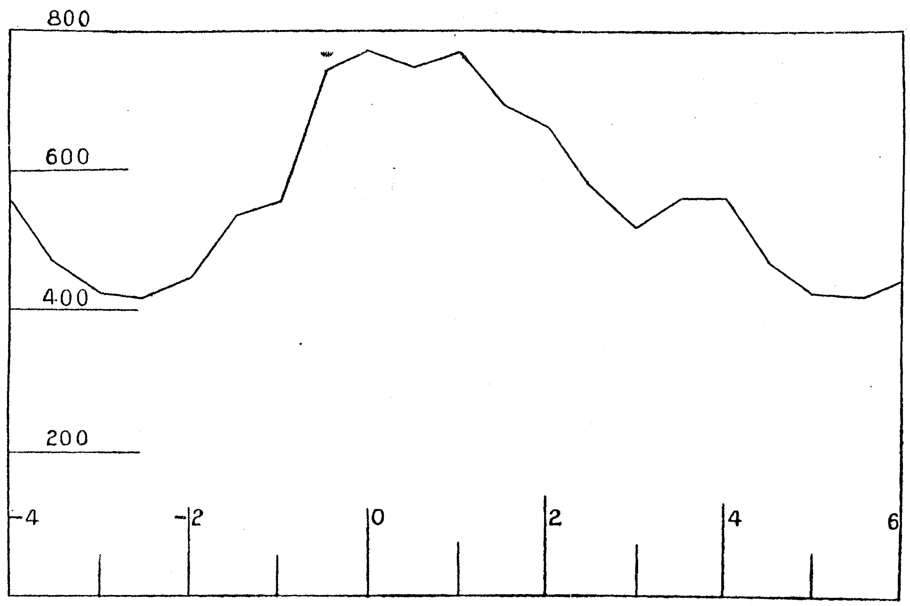

Fig. 7.

cannot be expected. We may take 206° as the most probable phase, which, converted into years, gives $1904{\cdot}51 \pm 8{\cdot}36n$ as the dates of the maxima of this period. It is significant that there are maxima with identical phases for the semi-period of 4·125

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years in the two portions of the time interval (see Table III.). This period wants further careful study. The rise and fall of the sunspot curve during the 8-year cycle is shown in fig. 7, which is based on WOLF'S numbers, making use of all observations since 1749.

16. We have now ascertained the existence of at least four periods, three of which have been determined with considerable accuracy. Their periodic times are 4·80, 8·36, and 11·125. Taking frequencies into consideration in place of periodic times, we are led to consider reciprocals and thus find

$$(11{\cdot}125)^{-1} = 0{\cdot}08989$$
$$(8{\cdot}36)^{-1} = 0{\cdot}11962$$
$$\text{Adding up, we find } (4{\cdot}77)^{-1} = 0{\cdot}20951$$

Hence the sum of the frequencies of two of the periods agrees within the limits of possible errors with the frequency of the third period. This relationship suggests an origin of the 4·8 years' variation, which may be similar to the origin of the vibrations of sound which go by the name of combination tones. Two causes varying as cos n_1t and cos n_2t respectively give rise, if the effect depends partly on the higher powers of the cause, to oscillations varying as cos $(n_1+n_2)t$ and cos $(n_1-n_2)t$. The corresponding time of the second combination tone would be 33·38 years.

But it is also found that the two first numbers are very nearly in the ratio of three to four, so that we may also express the three periodic times as sub-periods of 33·375 years, thus:

$$\tfrac{1}{3} \times 33{\cdot}375 = 11{\cdot}125$$
$$\tfrac{1}{4} \times 33{\cdot}375 = 8{\cdot}344$$
$$\tfrac{1}{7} \times 33{\cdot}375 = 4{\cdot}768$$

How far this connexion is accurate or approximate is impossible to say at present, but the fact that the three periods which have been traced with certainty should also bear a remarkably simple relationship to each other is worthy of note.

The question naturally occurs whether the great difference in sunspot activity observed when different maxima are compared with each other may be accounted for by the overlapping of different periods which sometimes will strengthen and sometimes weaken each other's effects. That the difference is not exclusively due to such overlapping is proved by the evidence of the periodogram, which for instance teaches us that in the second part of the eighteenth century the 11-year period was reduced to a fraction of its normal value.

Nevertheless it is true that in the exceptionally great outburst in 1870 nearly all the periods found seem to have been at their maximum phase. Similarly in the great outbreak of 1837 all but the 13·5-year period agreed in phase. This is shown by Table XVI.

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Table XVI.

Period.	Calculated maximum.	Calculated maximum.	Principal outbreaks.
4·38	1870·7	1835·7	1836·9
4·80	1870·4	1836·7	1837·5 1837·9
8·36	1871·1 [1869·2]	1837·6 [1836·1]	1870·2 1870·4
11·125	1870·4 1872·0	1837·0 1838·6	1870·7 1871·3
13·5	1870·0 [1871·4]	—	—

Of the two numbers given for the 11-year period the first refers to the observed maximum of the total oscillation which includes the harmonics, while the second gives the maximum phase of the fundamental period. The last column gives the dates of the principal outbreaks which together make up the two maxima concerned.

The significance of the table may be thought to be weakened to some extent by the consideration that the two exceptional maxima of 1837 and 1870 make themselves felt in the numerical reductions, tending to attract the maximum of any periodicity towards themselves. I have therefore for the two periods of 8·3 and 13·5 years added in brackets the dates of the maxima calculated from observations which all were previous to 1826, and which could not be affected in this manner.

17. The general conclusion arrived at brings us, I think, a decided step nearer to the solution of the problem of sunspot periodicities. The existence of a number of definite periods cannot be doubted, whatever we may think of their numerical relationship. The recurrence of the maximum activity of each period seems to take place with an accuracy which may be equal to that of orbital revolution, but the characteristic property of these periods is the great variability of the activity. To this variability we must attribute the cause why the periodicities have been allowed to remain hidden for so long a time. If we associate the periods, as seems natural, with the existence of meteor streams, these streams must vary in the power of setting up the disturbance which is the ultimate cause of the formation of a spot. I have for a long time believed in, and occasionally expressed, the opinion that sunspots are only secondary phenomena consequent on disturbances in the immediate surroundings of the sun, which are due to a general increase of electric conductivity affecting a great portion of interplanetary space. Such change in electric conductivity could be brought about by meteoric swarms fertilised by ionized matter which they have picked up in their journey through space.

The answer to many questions which naturally occur can only be given when further

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observations are available. But the existing material has not yet been exhausted, and may be expected to give further results, which I hope to present to the Society in due course.

TABLE I.—Comparison of Areas and WOLF'S Sunspot Numbers.

Year.	Mean daily area for each year.	WOLF'S sunspot numbers.	Ratio.	Year.	Mean daily area for each year.	WOLF'S sunspot numbers.	Ratio.
1832	274	27·5	9·96	1867	131	7·3	17·95
1833	60	8·5	7·06	1868	434	37·3	11·63
1834	133	13·2	10·08	1869	972	73·9	13·15
1835	773	56·9	13·58	1870	2761	139·1	19·36
1836	1331	121·5	10·96	1871	2004	111·2	18·02
1837	1170	138·3	8·46	1872	1462	101·7	14·38
1838	903	103·2	8·75	1873	766	66·3	11·55
1839	780	85·8	9·09	1874	601	44·7	13·45
1840	586	63·2	9·27	1875	272	17·1	15·91
1841	348	36·8	9·46	1876	122	11·3	10·80
1842	254	24·2	10·50	1877	92	12·3	7·48
1843	95	10·7	8·88	1878	19	3·4	5·59
1844	171	15·0	11·40	1879	30	6·0	5·00
1845	419	40·1	10·45	1880	402	32·3	12·45
1846	626	61·5	10·18	1881	679	54·3	12·50
1847	1021	98·5	10·36	1882	1002	59·7	16·78
1848	986	124·3	7·93	1883	1155	63·7	18·13
1849	765	95·9	7·98	1884	1079	63·5	16·99
1850	494	66·5	7·43	1885	811	52·2	15·53
1851	683	64·5	10·59	1886	381	25·4	15·00
1852	545	54·2	10·06	1887	179	13·1	13·67
1853	436	39·0	11·18	1888	89	6·8	13·09
1854	113	20·6	5·49	1889	78	6·3	12·38
1855	64	6·7	9·55	1890	99	7·1	13·94
1856	49	4·3	11·40	1891	569	35·6	15·99
1857	201	22·8	8·82	1892	1214	73·0	16·63
1858	714	54·8	13·03	1893	1464	84·9	17·25
1859	1404	93·8	14·96	1894	1282	78·0	16·44
1860	1172	95·7	12·25	1895	974	64·0	15·22
1861	1258	77·2	16·29	1896	543	41·8	12·99
1862	1363	59·1	23·06	1897	514	26·2	19·62
1863	702	44·0	15·95	1898	375	26·7	14·05
1864	784	47·0	16·68	1899	111	12·1	9·17
1865	407	30·5	13·35	1900	75	9·5	7·90
1866	318	16·3	19·51	1901	29	2·7	10·74

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Table IV.—Intensity and Phase of Periodogram.

I. Period in years.	II. 1750–1900.	III. 1750–1826.	IV. 1826–1900.	V. Beginning of second period.	VI. 1750–1900.	VII. 1750–1826.	VIII. 1826–1900.
24	48×10^3	42×10^3	13×10^3	1821	267°	259°	280°
23	17	59	3	1818	292	285	320
22	112	83	102	1815	239	288	208
21	349	198	191	1812	314	317	312
20	298	430	48	1809 (I)	4	352	32
19	35	147	174	1825	93	14	156
18	9	238	291	1821 (I)	295	42	235
17	83	224	38	1817	60	91	344
16	278	228	123	1813 (I)	164	186	139
15·5	340	304	91	1811 (I)	234	236	230
15·25	436	319	136	1825 (I)	238	238	237
15	434	394	117	1824 (I)	271	259	293
14·75	432	598	68	1822 (II)	296	282	342
14·5	342	755	6	1821 (II)	307	302	64
14·25	474	1058	36	1820 (I)	313	323	211
14	278	1122	40	1819 (I)	338	339	164
13·75	550	1337	6	1817 (II)	356	0	266
13·5	696	1170	33	1816 (II)	16	17	10
13·25	552	1150	77	1815 (I)	55	39	122
13	198	946	312	1814 (I)	100	57	191
12·75	21	788	517	1825 (II)	118	87	259
12·5	105	584	1019	1824 (I)	3	112	317
12·25	675	345	1526	1822 (II)	65	148	37
12	1464	289	2302	1821 (I)	125	185	107
11·86	1951	—	—	1820 (I)	155	—	—
11·75	2338	322	2894	1819 (II)	192	237	180
11·5	3700	465	3839	1818 (I)	253	265	250
11·25	4230	701	4396	1816 (II)	326	297	335
11	2724	605	4541	1826 (I)	31	325	50
10·75	742	793	3774	1824 (I)	122	349	142
10·5	853	877	3202	1822 (II)	251	356	221
10·25	2026	1179	1842	1820 (II)	337	16	306
10	1677	1475	440	1819 (I)	27	22	35
9·75	1050	1818	84	1817 (I)	60	49	107
9·5	1313	2017	156	1825 (I)	75	62	129
9·25	603	2050	440	1823 (I)	120	94	226
9	364	1885	257	1821 (I)	97	114	322
8·75	812	1390	142	1819 (I)	135	153	282
8·5	770	743	185	1825 (II)	163	174	140
8·25	933	486	495	1823 (I)	241	228	254
8	177	374	404	1821 (I)	336	266	30
7·75	—	402	269	1811 (I)	—	334	201
7·5	264	371	22	1824 (I)	18	25	345
7·375	173	—	—	—	63	—	—
7·25	106	264	34	—	54	74	318
7·125	149	—	—	—	94	—	—
7	99	68	82	—	158	116	188
6·875	18	—	—	—	317	—	—
6·75	52	7	61	—	55	94	44
6·625	49	—	—	—	124	—	—
6·5	1	24	25	—	192	64	231
6·375	7	—	—	—	145	—	—

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Table IV.—Intensity and Phase of Periodogram (continued).

I. Period in years.	II. 1750–1900.	III. 1750–1826.	IV. 1826–1900.	V. Beginning of second period.	VI. 1750–1900.	VII. 1750–1826.	VIII. 1826–1900.
6·25	5×10^3	23×10^3	13×10^3	—	28°	76°	295°
6·125	24	—	—	—	62	—	—
6	102	24	136	—	154	105	173
5·93	54	—	—	—	202	—	—
5·875	122	—	—	—	293	—	—
5·75	236	3	397	—	63	346	68
5·625	128	55	420	—	249	35	239
5·5	224	62	259	—	37	78	18
5·375	129	—	—	—	160	—	—
5·25	15	28	33	—	238	174	295
5·125	31	—	—	—	286	—	—
5	12	44	2	—	305	310	147
4·875	97	152	14	—	7	21	321
4·75	99	170	49	—	86	56	153
4·625	70	132	8	—	108	122	33
4·5	25	56	2	—	159	173	79
4·375	41	41	19	—	305	278	341
4·25	9	59	14	—	355	341	145
4·125	109	81	34	—	54	57	50
4	29	57	33	—	25	73	323
3·875	—	12	6	—	—	0	227
3·75	24	14	41	—	89	22	121
3·67	—	—	61	—	—	—	5
3	—	1	2	—	—	—	—

Table V.—Intensity of Periodogram 5·18–1·89 years.

I. Period in years.	II. Intensity of periodogram.	I. Period in years.	II. Intensity of periodogram.	I. Period in years.	II. Intensity of periodogram.
5·18	23×10^3	3·78	121×10^3	2·49	1×10^3
5·08	10	3·69	75	2·44	13
4·98	51	3·59	18	2·39	21
4·88	133	3·49	22	2·34	14
4·78	167	3·39	13	2·29	19
4·68	105	3·29	5	2·24	17
4·58	15	3·19	6	2·19	26
4·48	47	3·09	29	2·14	39
4·38	123	2·99	1	2·09	21
4·28	79	2·89	16	2·04	30
4·18	29	2·79	46	1·99	16·1
4·08	43	2·69	49	1·94	54·2
3·98	7	2·59	43	1·89	25·3
3·88	39	2·54	11		

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TABLE VI.—Intensity of Periodogram for Periods of 2·240 years to 55 days.

Period.	Intensity.	Mean intensity.	Period.	Intensity.	Mean intensity.	Period.	Intensity.	Mean intensity.
Years.			Years.			Days.		
2·240	87	310×10^2	1·120	81	137×10^2	136	523	163×10^2
	72			24			12	
	1059			432			34	
	30			10			16	
2·090	53	290	1·045	29	198		366	
	17			44			27	
	832			499		116	27	45
	257		Days.	218		(Mercury)	19	
1·941	17	230	354·5	11	181		142	
	194			343			6	
	583			204			42	
	127			168			32	
1·792	106	131	327·24	423	367	109	366	116
	221			561			27	
	160			291			108	
	38			192			6	
1·642	113	204	299·97	409	188		99	
	99			154			90	
	141			65		68	172	90
	463			124			15	
1·599	297	240	291·92	109	42		106	
(Venus)	171			26			10	
	27			2			138	
	464			33			93	
1·568	159	132	286·33	2	63	58	192	81
	127			117			12	
	60			103			130	
	180			32			19	
1·493	48	120	272·7	8	167		0	
	205			291			129	
	91			338		55	1006	181
	138			33			1	
1·344	359	185	245·4	124	147		12	
	7			96			15	
	304			309			42	
	69			58			10	
1·195	355	199	218·16	7	179			
	45			126				
	175			579				
	223			6				

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TABLE VII.—Intensity of Periodogram for Periods of 30–13·3 days.

Period in days.	1886–1890.	1890–1895.	1895–1900.	Mean.	General average.
30	840	8	13,560	4,800	
29	830	12,970	15,240	9,680	12,400
28	600	64,470	3,050	22,710	
27·27	7,640	21,930	410	9,990	
27	2,210	50,610	2,550	18,460	10,240
26·5	2,990	1,800	2,000	2,260	
26	1,430	83,150	3,860	29,480	
25	240	6,370	4,640	3,750	11,440
24	1,130	670	1,440	1,080	
15	170	1,500	220	630	
14·5	280	1,150	570	670	520
14	100	380	270	250	
13·635	300	850	70	410	
13·5	270	2,090	50	800	1,210
13·25	100	5,850	1,320	2,430	
13	100	480	710	430	
12·5	50	3	760	270	360
12	240	30	850	370	
13·298	—	—	—	3,080	
13·296	—	—	—	2,150	

TABLE VIII.—Intensity of Periodogram, Values of A and B, and Phases of the 26- and 27-day Periods for Separate Years.

Year.	Period of 26 days.	Period of 27 days.	A.		B.		ϕ.	
			26 days.	27 days.	26 days.	27 days.	26 days.	27 days.
1886	376×10^2	856×10^2	−24627	−40513	−12679	+12973	207°	162°
1887	157	169	+ 5292	−16363	+17068	+ 8062	73	154
1888	26	17	+ 1	+ 5934	+ 7222	− 128	90	359
1889	75	58	− 3426	+ 7075	−11855	− 8062	254	311
1890	36	82	− 7821	+10934	− 3535	− 7274	204	326
1891	1024	994	+12457	+16437	−43946	−41146	286	292
1892	1376	407	+29666	−29218	−43865	+ 2340	304	175
1893	1202	2899	+ 4419	+ 4639	−49310	−75526	275	274
1894	2854	1445	+74603	−54639	+15907	− 8310	12	189
1895	2007	3645	+37343	+60644	−51933	−59345	306	316
1896	62	279	−11514	+11137	+ 1570	−21563	172	297
1897	268	372	−22851	+13126	− 4903	+23705	192	61
1898	901	144	+ 1208	+ 333	+42845	−17454	88	271
1899	13	14	+ 3968	+ 4230	− 3408	− 3062	319	324
1900	67	53	+ 6396	+ 902	+ 9800	−10532	57	275